Graph Theory

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1 Terms and Theorems

Terminology:

- A graph G is a pair (V, E) where V is a set of vertices and E is a set of edges which are unordered pairs of vertices. In a directed graph, E is a set of ordered pairs of vertices.
- An edge $\{u, v\}$ is said to be incident to its two endpoints u and v and two vertices $u, v \in V$ are said to be adjacent if $\{u, v\} \in E$.
- The degree $\deg(u)$ of a vertex u is the number of $v \in V$ such that u is adjacent to v.
- A path is a sequence $v_1, v_2, \ldots, v_k$ of distinct vertices such that $\{v_i, v_{i+1}\} \in E$ for all $1 \leq i \leq k-1$. A cycle also satisfies that $\{v_k, v_1\} \in E$. A walk is path and a circuit is a cycle with the relaxed condition that vertices can be repeated and a circuit is a cycle.
- A graph is connected if there is a path between every pair of vertices. A tree is a graph that is connected and contains no cycle.
- The complete graph K_n with n vertices satisfies that all pairs of vertices are adjacent.
- A graph is bipartite if its vertices can be partitioned into two sets A and B such that all edges in the graph join vertices in A to vertices in B.
- A complete bipartite graph $K_{m,n}$ is a bipartite graph with $|A| = m$, $|B| = n$ and such that all pairs of vertices between A and B are joined by an edge.
- A graph is planar if it can be drawn in a plane in such a way that its edges only intersect at their endpoints where its edges are represented as curved lines.
- A subset of vertices in a graph is independent if no two vertices are joined by an edge.

Theorems:

1. (Handshaking Lemma) Given a graph $G = (V, E)$, the sum of the degrees of the vertices in V is even with
$$\sum_{v \in V} \deg(v) = 2 \cdot |E|$$
2. If a graph G is connected and contains $n-1$ edges, then it is a tree.

3. If a graph G with n vertices is connected, then it has at least $n-1$ edges and there is a *spanning tree* T which contains all vertices and a subset of the edges of G.

4. A graph G is bipartite if and only if it contains no cycles of odd length.

5. (Euler's Characteristic) A planar graph with V vertices, F faces and E edges satisfies

$$V + F = E + 2$$

6. (Hall's Theorem) Let $G = (V, E)$ be a bipartite graph with parts A and B where $V = A \cup B$. Given a subset $S \subseteq A$, let $\Gamma(S)$ denote the set of neighbors of S in B. Each vertex in A can be matched with a unique vertex in B if and only if for all S,

$$|\Gamma(S)| \geq |S|$$

7. (Eulerian Circuit) Given a graph G, there is an *Eulerian walk* passing through each edge exactly once if and only there are at most two vertices of odd degree. There is an *Eulerian circuit* passing through each edge exactly once if and only if all vertices have even degree. If G is directed, there is an Eulerian circuit if and only if the in-degree of each vertex equals its out-degree.

8. (Kuratowski's Theorem) A graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$.

9. (Turan's Theorem) Let G be a graph that contains no complete subgraph with k vertices. The number of edges e of G satisfies that

$$e \leq \frac{k-2}{k-1} \cdot \frac{n^2 - r^2}{2} + \binom{r}{2}$$

where $r < k-1$ and $n \equiv r \pmod{k-1}$.

10. (Ramsey's Theorem) Given two positive integers r and s, there is a minimal positive integer $R(r, s)$ such that any graph with at least $R(r, s)$ vertices either contains a complete subgraph with r vertices or an independent set with s vertices. It holds that

$$R(r, s) \leq \binom{r+s-2}{r-1} \quad \text{and} \quad R(r, s) \leq R(r-1, s) + R(r, s-1)$$

2 Examples

The first example illustrates a case in which the underlying graph is not immediately obvious but, when found, makes the problem much more tractable. The idea is also commonly used to analyze square grids. The squares in the grid can be viewed as edges in a bipartite graph in which vertices correspond to the rows and columns of the grid.

Example 1. *(IMC 1999) Suppose* $2n$ *squares of an* $n \times n$ *grid are marked. Prove that there exists a* $k > 1$ *and* $2k$ *distinct marked squares* $a_1, a_2, \dots, a_{2k}$ *such that for all* i*,* a_{2i-1} *and* a_{2i} *are in the same row while* a_{2i} *and* a_{2i+1} *are in the same column where* $a_{2k+1} = a_1$*.*

Solution. Consider the bipartite graph with vertices $V = A \cup B$ where the vertices in A correspond to the columns and the vertices in B correspond to the rows of the $n \times n$ grid. Each marked square is represented by an edge between the vertices for its row and column and the number of edges is $2n$. If the bipartite graph does not contain a cycle, it is the disjoint union of t trees and the number of edges is $|V| - t = 2n - t < 2n$. This is a contradiction and the graph must contain a cycle. Since the graph is bipartite, the cycle has even length $2k$. Consecutive edges in the cycle share an endpoint and therefore alternate between sharing a row or column. Therefore the marked squares corresponding to this cycle are the desired marked squares $a_1, a_2, \dots, a_{2k}$.

The second example is a simple but classical use of induction to prove a general result on graphs. Many graph theory problems either have an inductive solution or rely on a lemma or theorem that can be shown inductively.

Example 2. *There are* n *teams in a tennis tournament and every pair of teams faces off in a single match. If there are no ties, prove that the teams can be arranged in a line so that each team beats the team to its left.*

Solution. We write $a \to b$ if team b beat team a. We prove the claim by induction. The base case when $n = 1$ is clearly true. Now assume the claim is true for $n - 1$ and consider a tournament G with n teams. Consider an arbitrary team u and the sub-tournament of G not involving u. By the induction hypothesis, these $n - 1$ teams can be placed in an order such that $v_1 \to v_2 \to \cdots \to v_{n-1}$. If $u \to v_1$, then we may append u to the front to yield a valid ordering. Therefore we can assume that $v_1 \to u$. Similarly if $u \to v_2$, then u may be placed between v_1 and v_2 to give a valid ordering and we may assume that $v_2 \to u$. Repeating this reasoning yields that unless $v_1 \to u, v_2 \to u, \dots, v_{n-1} \to u$, u can be placed in the list to form a valid ordering. But in this case, u can be placed at the end of the line. In any case, this completes the induction and proves the claim.

The next two examples illustrate how analyzing subgraphs or different representations of a graph can solve problems. In the examples below, a subgraph that is the disjoint union of cycles is analyzed. In other cases, looking at subgraphs such as spanning trees or independent sets in the graph can be useful.

Example 3. *(Caroll) A building consists of* 4004001 *rooms arranged in a* 2001×2001 *square grid. Is it possible for each room to have exactly two doors to adjacent rooms?*

Solution. The answer is no. Consider the graph G such that the vertices of G are the rooms in the building and two vertices are joined by an edge if there is a door between them. If each room has exactly two doors, then each vertex of G has degree two. It will be shown that this implies that G is the disjoint union of cycles. Begin with a vertex v_1 and pick $v_2, v_3, \dots$ iteratively such that v_i is adjacent to v_{i+1} and $v_{i-1} \neq v_{i+1}$ for all $i \geq 2$. Note that this is possible since each vertex has degree two. Since G has a finite number of vertices, there is a v_k in the sequence of vertices such that v_k has appeared previously. Let k be the minimal positive integer for which this is true. It follows that $v_1, v_2, \dots v_{k-1}$ are distinct vertices. If $v_k = v_i$ where $2 \leq i \leq k-2$, then v_i is adjacent to v_{i-1}, v_{i+1} and v_{k-1} which are distinct vertices, which contradicts $\deg(v_i) = 2$. Therefore $v_1 = v_k$ and $v_1, v_2, \dots, v_{k-1}$ form a cycle. Also if T is the set of vertices of G other than $v_1, v_2, \dots, v_{k-1}$, then no vertex in T can be adjacent to any of $v_1, v_2, \dots, v_{k-1}$ since otherwise $\deg(v_i) > 2$ would be true for some $1 \leq i \leq k-1$. Repeating this argument on T decomposes G into disjoint cycles. Now note that every cycle of G consists of an even number of doors between rooms in the same column and an even number of doors between rooms in the same row since the cycle must return to its start point. Thus every cycle is even and G must have an even number of vertices since it is the disjoint union of cycles. This contradicts the fact that there are 4004001 rooms.

Example 4. *(Russia 1999) In a country, there are N airlines that offer two-way flights between pairs of cities. Each airline offers exactly one flight from each city in such a way that it is possible to travel between any two cities in the country through a sequence of flights, possibly from more than one airline. If $N-1$ flights are cancelled, all from different airlines, show that it is still possible to travel between any two cities.*

Solution. Let G denote the corresponding graph and G' denote the graph with the $N-1$ flights removed. Let C be the airline with no flights cancelled and let $C_1, C_2, \dots, C_{N-1}$ denote the other $N-1$ airlines. Let G_k denote the subgraph with flights offered by either C or C_k. Each city is the endpoint of exactly one flight offered by C and one offered by C_k. Therefore each city has degree exactly two in G_k and, as proven in the example above, this implies G_k is the disjoint union of cycles. Therefore if e is the flight offered by airline C_k that is cancelled, then e is the only edge from G_k not in G'. Also, e is in a cycle in G_k and the other edges in this cycle form a path between the endpoints of e in G'. Therefore for every flight e cancelled, the endpoints of e are still connected in G' implying G' is connected.

The next example is a theorem from Dirac that illustrates the method of looking at an extremal part of a graph. In this case, the longest path is examined to show the existence of a Hamiltonian cycle.

Example 5. *(Dirac) Given a graph G with n vertices such that each vertex has degree at least $\frac{n}{2}$, prove that G has a Hamiltonian cycle.*

Solution. First it will be shown that G is connected. If G were not connected and C is the smallest connected component, then $|C| \leq \frac{n}{2}$ and each vertex in C has degree at most

$|C| - 1 < \frac{n}{2}$, which is a contradiction. Thus G is connected. Now consider the longest path $v_1, v_2, \dots, v_k$ in G. If v_1 or v_k is adjacent to any vertex u not in the path, then the path can be elongated by appending u to one of its ends, which is a contradiction. Now it will be shown that there is an i with $1 \le i \le k-1$ and such that $v_1 v_i$ and $v_{i+1} v_k$ are both edges in G. If not, then for each i, at most one of $v_1 v_i$ and $v_{i+1} v_k$ are edges in G, implying that there are at most $k-1$ edges adjacent to either v_1 or v_k in the path. Since all neighbors of v_1 and v_k are in the path, this implies that one of v_1 or v_k has degree at most $\frac{k-1}{2} < \frac{n}{2}$, which is a contradiction. Thus the vertices $v_1, v_2, \dots, v_i, v_k, v_{k-1}, \dots, v_{i+1}$ form a cycle. Assume for contradiction that there is some vertex u not on this cycle. Since G is connected, u is adjacent to v_i for some i and there is a path of length $k+1$ beginning at u followed by the vertices on the cycle beginning with v_i, contradicting the maximality of the original path. Thus the cycle is a Hamiltonian cycle.

The last example is a lemma with a slightly more computational proof than those above. This result can be useful in problems concerning k-free graphs, those with no complete subgraph with k vertices.

Example 6. *(Zarankiewicz) Let G be a graph with n vertices that contains no complete subgraph with k vertices. Prove that there exists a vertex of degree at most $\lfloor \frac{n(k-2)}{k-1} \rfloor$.*

Solution. Assume that every vertex has degree greater than $\frac{n(k-2)}{k-1}$. Begin by setting $T = \{v\}$ where v is an arbitrary vertex of G and let S be the set of vertices adjacent to all vertices in T. It follows that initially $|S| > \frac{n(k-2)}{k-1}$. Append a vertex u in S to T. Now

$$|S| > 2 \cdot \frac{n(k-2)}{k-1} - N > 2 \cdot \frac{n(k-2)}{k-1} - n$$

where N is the number of vertices adjacent to both u and v. Continuing this process until $|T| = j$ yields similarly that

$$|S| > j \cdot \frac{n(k-2)}{k-1} - (j-1)n$$

If $j \le k-1$, it follows that $|S| > 0$ and thus another vertex can be appended to T. Therefore this process continues until $|T| \ge k$. Note that this process ensures T is complete subgraph with $|T|$ vertices. Thus T eventually contains a complete subgraph with k vertices. This is a contradiction and the desired result holds.

3 Problems

1. (Erdos) Given a graph G with n vertices such that each vertex has degree at least k, prove that G has a cycle of length $k+1$.

2. Show that every graph G with average degree d has a subgraph in which every vertex has degree at least $d/2$.

3. (HMMT 2003) There are a people who want to share b applies so that they all get equal quantities of apple where $a > b$. Prove that at least $a - \gcd(a, b)$ cuts are required.

4. (Russia 2003) There are N cities in a country. Any two of them are connected either by a road or by an airway. A tourist wants to visit every city exactly once and return to the city at which he started the trip. Prove that he can choose a starting city and make a path, changing means of transportation at most once.

5. (Bulgarian MO 2004) A group consists of n tourists. Among any three of them, there are two who do not know each other. For every partition of the tourists into two buses, we can always find two tourists in the same bus who know each other. Prove that there is a tourist who knows at most $\frac{2n}{5}$ other tourists.

6. A county contains 2010 cities, some pairs of which are linked by roads. Show that the country can be divided into two states S and T such that each state contains 1005 cities and at least half the roads connect a city in S with a city in T.

7. (IMO Shortlist 1983) Let n be a positive integer. Suppose that n airline companies offer trips to citizens of N cities such that for any two cities there exists exactly one direct flight between the two cities. Find least N such that we can always find a company that offers a trip in a cycle with an odd number of landing points.

8. Prove that one can write 2^n numbers around a circle, each equal to 0 or 1 so that any string of n 0's and 1's can be obtained by starting somewhere on the circle and reading the next n digits in clockwise order.

9. (IMO Shortlist 2012) The columns and the row of a $3n \times 3n$ square board are numbered $1, 2, \ldots, 3n$. Every square (x, y) with $1 \leq x, y \leq 3n$ is colored red, blue or green according as the modulo 3 remainder of $x + y$ is $0, 1$ or 2 respectively. One token colored red, blue or green is placed on each square, so that there are $3n^2$ tokens of each color. Suppose that on can permute the tokens so that each token is moved to a distance of at most d from its original position, each red token replaces a blue token, each blue token replaces a green token, and each green token replaces an red token. Prove that it is possible to permute the tokens so that each token is moved to a distance of at most $d + 2$ from its original position, and each square contains a token with the same color as the square.

10. (IMO Shortlist 2002) Let n be an even positive integer. Prove that there exists a permutation $x_1, x_2, \ldots, x_n$ of $1, 2, \ldots, n$ such that x_{k+1} is equal to one of $2x_k, 2x_k - n, 2x_k - 1$, or $2x_k - n - 1$ for all $1 \leq k \leq n$ where $x_{n+1} = x_1$.

11. (Russia 1997) Let m and n be given odd positive integers. An $m \times n$ grid is covered by dominoes so that exactly one corner square remains uncovered. A move consists of sliding a domino into the empty square. Show that any corner square can be made empty through a finite sequence of moves.

12. (Iranian TST 2006) Given a complete directed G with each edge colored either red or blue, prove that there is a vertex v such that for all other vertices u there is a directed path from v to u using edges of a single color.

13. (Chinese TST 1997) A graph with $n^2 + 1$ edges and $2n$ vertices is given. Prove that it contains two triangles that share a common edge.

14. A graph G has n vertices and m edges. If the edges are assigned the labels $1, 2, \ldots, m$, prove that there exists a path consisting of at least $\frac{2m}{n}$ vertices such that the labels of the edges along the path are in increasing order.