1 Introduction

These notes are about geometry. But geometry is a huge subject, and I am only going to be scratching the surface here. If you are serious about doing well in Olympiads, I strongly recommend seeking out more resources on your own. Geometry Unbound by Kiran Kedlaya is a great starting point:

http://www-math.mit.edu/~kedlaya/geometryunbound/

Previous notes from Canadian winter and summer camps have been less encyclopedic, but they are exceptionally good. Take advantage of them too!

http://sites.google.com/site/imocanada/

2 Configurations and Directed Angles

All right, down to business! Instead of starting with a technique, I am going to start with a pitfall. What do the following examples have in common?

Example 1. Let $ABCD$ be a cyclic quadrilateral. The perpendiculars to $AD$ and $BC$ at $A$ and $C$ respectively meet at $M$, and the perpendiculars to $AD$ and $BC$ at $D$ and $B$ meet at $N$. If the lines $AD$ and $BC$ meet at $E$, prove that $\angle DEN = \angle CEM$.

“Solution”: Since $\angle EBN = \angle EDN = 90^\circ$, $EBDN$ is a cyclic quadrilateral. $EACM$ is also cyclic for the same reasons. Now:

$$\angle DEN = 180^\circ - \angle BED - \angle BDN$$

$$= 90^\circ - \angle BDA$$

$$= 90^\circ - \angle AEC - \angle ACM$$

$$= 180^\circ - \angle AEC - \angle ACM$$

$$= \angle CEM,$$

and the problem is solved.  

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1Iran 2004, and Winter Camp 2011 Warmup Problem.
Example 2. Let $ABC$ be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides $BC$ and $CA$ at $D$ and $E$ respectively. Let $K$ be the incenter of triangle $ADC$. Suppose that $\angle BEK = 45^\circ$. Find all possible values of $\angle CAB$.

“Solution”: Let $I$ be the incenter of $\triangle ABC$. Note that $K$ and $I$ both lie on the angle bisector of $\angle ACB$. Let $E'$ be the reflection of $E$ across line $CKI$. $E'$ lies on $BC$ do you see why?

$\angle KE'I = \angle KEI = 45^\circ$, and furthermore, $\angle CDA = 90^\circ$ since $AB = AC$ and $AD$ is an angle bisector. Because $K$ is the incenter of $\triangle ADC$, we have $\angle KDI = \frac{1}{2} \cdot \angle CDA = 45^\circ = \angle KE'I$. Therefore, $KE'DI$ is a cyclic quadrilateral.

Now let $\angle CAB = \theta$, so that $\angle ECB = 90 - \frac{\theta}{2}$, and $\angle EBC = 45 - \frac{\theta}{4}$. Since $\angle BEK$ is given to be $45^\circ$ and the sum of the angles in $\triangle ECB$ is $180^\circ$, we have $\angle CEK = \frac{360}{4}$. On the other hand, since $KE'DI$ is cyclic, we know $\angle CEK = \angle CE'K = \angle CID = 90^\circ - \angle ICD = 45 + \frac{\theta}{4}$. Equating our two expressions for $\theta$, we are left with $\theta = 90^\circ$.

Example 3. Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game: Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex. Show that the player who signs first will always win by playing as well as possible.

“Solution”: Suppose the polyhedron has a face $A$ with at least 4 edges. If the first player begins by signing there, then after the second player’s turn, there will be 3 consecutive faces $B, C, D$ adjacent to $A$, which are all unoccupied. The first player wins by signing $C$; after the second player’s second move, at least one of $B$ or $D$ remains unoccupied, and either is a winning move for the first player.

It remains to show that the polyhedron has a face with at least four edges. Suppose on the contrary that each face has only three edges. Start with any face having vertices $a, b, c$. Let $d$ be the third vertex that is adjacent to $a$. Then $a, b, d$ and $a, c, d$ must be faces, and then finally, $b, c, d$ must also be a face. But we have now completed the polyhedron, and it had only 4 faces, which is a contradiction. Therefore, one face must have at least 4 edges, and the proof is complete.

So what do all these “solutions” have in common? The answer is they are all wrong!

- Actually, Example 1 is just incomplete. The proof neglects to mention the case where $M$ is on the opposite side of $CE$, and also the case where $N$ is the opposite side of $DE$. If $M$ and $N$ are both flipped, you can still solve the problem fine. If only one is flipped however, the argument fails. Good thing that case is impossible! But you really need to show that...

- In Example 2, does anything change if $E'$ lies on the opposite side of $D$? What if $E'$ is the same as $D$? Hint: $\theta = 90^\circ$ is not the only possible answer! The solution here would have gotten only 4/7. It missed an answer, and it failed to verify that $\theta = 90^\circ$ really does work.

\[ ^2 \text{IMO 2009, #4.} \]
\[ ^3 \text{Putnam 2002, B2.} \]
• In Example 3, the whole problem is wrong! Start with a tetrahedron, and consider removing a triangular chunk from one edge, as shown above. In this configuration, two faces share two different edges, and the whole proof breaks down. In fact, the first player cannot win here at all. Do you see why?

These examples illustrate the importance of making sure your argument works for all configurations, not just the one you drew! In particular, when you use angle-chasing, you should expect configuration issues like in Example 1. Most problems won’t be as subtle as these ones were, but having two similar cases is extremely common. If you ignore some of the configurations, you are just asking to lose points.

Here is one useful (but limited) way of dealing with multiple configurations while angle chasing:

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**Directed Angles mod 180°:**

Given two lines \( \ell_1 \) and \( \ell_2 \), we define the directed angle between them: \( \angle(\ell_1, \ell_2) \) to be the counter-clockwise angle that you need to rotate \( \ell_1 \) by in order to make it parallel to \( \ell_2 \). Note that 180° is the same as 0° in directed angles: if you rotate a line by 180°, it is still parallel to itself.

The following statements about directed angles are true, regardless of configuration:

1. \( \angle(\ell_1, \ell_2) = -\angle(\ell_2, \ell_1) \)
2. \( \angle(\ell_1, \ell_3) = \angle(\ell_1, \ell_2) + \angle(\ell_2, \ell_3) \)
3. \( \angle(\ell_1, \ell_2) + \angle(\ell_2, \ell_3) + \angle(\ell_3, \ell_1) = 0 \)
4. \( A, B, C, D \) lie on a circle iff \( \angle(AB, BC) = \angle(AD, DC) \)
5. \( B, C, D \) lie on a line iff \( \angle(AB, BC) = \angle(AB, BD) \)
6. \( AB \) is tangent to the circle through \( B, C, D \) iff \( \angle(AB, BC) = \angle(BD, DC) \)
7. If \( A, B, C \) lie on a circle with center \( O \), then \( 2\angle(AB, BC) = \angle(AO, OC) \)

Properties 4,5,6 are what make directed angles useful. With regular angles, there are always two cases based upon the ordering of the points: the angles might be equal or supplementary. Directed angles remove the cases altogether, which makes them perfect for showing collinearity or cyclicity. They are usually NOT a good choice for showing two regular angles are equal however: even if two directed angles are equal, the regular angles could be supplementary instead of equal!
Example 4. Suppose that the circles \( \omega_1 \) and \( \omega_2 \) intersect at distinct points \( A \) and \( B \). Let \( CD \) be any chord on \( \omega_1 \), and let \( E \) and \( F \) be the second intersections of the lines \( CA \) and \( BD \), respectively, with \( \omega_2 \). Prove \( EF \) is parallel to \( DC \).

Solution: There are many, many possible configurations here, making regular angle-chasing impractical. It’s a breeze with directed angles though:

\[
\angle(\ EF, DC \) = \angle(\ EF, DBF) + \angle(\ DBF, DC) = \angle(\ EF, FB) + \angle(\ BD, DC) = \angle(\ EA, AB) + \angle(\ BA, AC) = \angle(\ EA, AC) = 0,
\]

and hence \( EF \) and \( DC \) are indeed parallel.

Final warning: While directed angles are very useful in the right situation, there are a couple pitfalls you have to watch out for. I already mentioned that two directed angles being equal does not imply that the corresponding regular angles are equal. Here is another one: Do not ever use expressions like \( \frac{1}{2} \cdot \angle(AB, BC) \), because half of a directed angle has no meaning!

For example, in the diagram to the right, \( 2 \cdot \angle(AB, BP) = \angle(AB, BC) = 2 \cdot \angle(AB, BQ) \), but \( \angle(AB, BP) \neq \angle(AB, BQ) \). In other words, there are two possible angles that you could mean when you say \( \frac{1}{2} \cdot \angle ABC \). So how do you deal with this? The answer is: never, ever look at a fraction of an angle. Modular arithmetic in number theory has the same issue.

Exercises:

1. Convince yourself that the listed properties for directed angles are true.
2. Correct the proof for Example 1.
3. Correct the proof for Example 2.
4. Fix triangle \( ABC \), and choose points \( P, Q, R \) on lines \( BC, CA \), and \( AB \), respectively. Prove that the circumcircles of triangles \( AQR, BRP \), and \( CPQ \) pass through a common point.
5. Let \( \omega_1, \omega_2, \omega_3, \omega_4 \) be four circles in the plane. Suppose that \( \omega_1 \) and \( \omega_2 \) intersect at \( P_1 \) and \( Q_1 \), \( \omega_2 \) and \( \omega_3 \) intersect at \( P_2 \) and \( Q_2 \), \( \omega_3 \) and \( \omega_4 \) intersect at \( P_3 \) and \( Q_3 \), and \( \omega_4 \) and \( \omega_1 \) intersect at \( P_4 \) and \( Q_4 \). Show that if \( P_1, P_2, P_3, \) and \( P_4 \) lie on a circle, then \( Q_1, Q_2, Q_3, \) and \( Q_4 \) also lie on a circle.
6. Triangle \( ABC \) is any one of the set of triangles having base \( BC \) equal to \( a \) and height from \( A \) to \( BC \) equal to \( h \), with \( h < \frac{\sqrt{3}}{2} \cdot a \). \( P \) is a point inside the triangle such that \( \angle PAB = \angle PBA = \angle PCB = \alpha \). Show that the measure of \( \alpha \) is the same for every triangle in the set.
7. Two circles intersect at points $A$ and $B$. An arbitrary line through $B$ intersects the first circle again at $C$ and the second circle again at $D$. The tangents to the first circle at $C$ and the second at $D$ intersect at $M$. Through the intersection of $AM$ and $CD$, there passes a line parallel to $CM$ and intersecting $AC$ at $K$. Prove that $BK$ is tangent to the second circle.

8. Determine all finite sets $S$ of at least three points in the plane which satisfy the following condition: For any two distinct points $A$ and $B$ in $S$, the perpendicular bisector of the segment $AB$ is an axis of symmetry for $S$.

3 Power of a Point

Angle-chasing – especially to find cyclic quadrilaterals and congruent triangles – is the most important technique in all of Olympiad geometry. It doesn't always work though, and one of the simplest and most reliable alternatives to try next is power of a point.

Fact 1. Suppose lines $AB$ and $CD$ meet at a point $P$. Then $A, B, C, D$ all lie on a circle if and only if the "directed lengths" $PA, PB, PC, PD$ satisfy $PA \cdot PB = PC \cdot PD$.

Fact 2. Suppose point $P$ lies on line $AB$, and $C$ is an arbitrary point. Then, $PC$ is tangent to the circle through $A, B, C$ if and only if the directed lengths $PA, PB, PC$ satisfy $PA \cdot PB = PC^2$.

So first of all: what are directed lengths? The important thing is $PA \cdot PB$ is considered negative if $P$ is between $A$ and $B$, and positive otherwise.

Hopefully, you have seen these facts before. They are already powerful enough to solve some otherwise difficult problems:

Example 5. Let $C_1$ and $C_2$ be concentric circles, with $C_2$ in the interior of $C_1$. From a point $A$ on $C_1$, one draws the tangent $AB$ to $C_2$ ($B \in C_2$). Let $C$ be the second point of intersection of $AB$ and $C_1$, and let $D$ be the midpoint of $AB$. A line passing through $A$ intersects $C_2$ at $E$ and $F$ in such a way that the perpendicular bisectors of $DE$ and $CF$ intersect at a point $M$ on $AB$. Find, with proof, the ratio $AM/MC$.

Solution: If $R_1$ and $R_2$ are the radii of $C_1$ and $C_2$, then $AB^2 = R_1^2 - R_2^2 = BC^2$, so $B$ is the midpoint of $AC$. Therefore,

$$AD \cdot AC = \left(\frac{AB}{2}\right) \cdot (2AB) = AB^2 = AE \cdot AF$$

since $AB$ is tangent to $C_2$,

which implies $DCFE$ is cyclic. The center of this circle lies on the perpendicular bisectors of $DE$ and $CF$, which means it must be $M$. Therefore, $MD = MC$, and hence $MC = \frac{1}{2} \cdot DC = \frac{3}{8} \cdot AC$. Then, $AM/MC = \frac{5}{3}$.

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4Here is the proper definition if you are interested: Assign each direction on the plane to be positive or negative in any fashion you want, ensuring only that if two directions are opposite each other, then they have opposite signs. Then a length $XY$ is considered positive if $XY$ points in a positive direction, and negative otherwise. If $X, Y, Z$ are collinear, this ensures that $XY + YZ = XZ$ no matter what order they lie on, giving directed lengths many of the same advantages as directed angles. Ceva’s theorem and Menelaos’s theorem are properly stated using directed lengths as well.

5USAMO 1998, #2.
Power of a Point:

Fix a point $P$ and a circle $\omega$ with center $O$ and radius $r$. Let $\ell$ be any line through $P$ intersecting $\omega$ at $A$ and $B$. By Fact 1, the product of directed lengths $PA \cdot PB$ does not depend on $\ell$. This value is called the power of $P$ with respect to $\omega$.

Bonus Fact: The power of $P$ with respect to $\omega$ is equal to $OP^2 - r^2$.

While Facts 1 and 2 are certainly handy, the main use of power of a point comes from the radical axis:

Radical Axis:

Fix two circles $\omega_1$ and $\omega_2$ with different centers. The set of points that have equal power with respect to $\omega_1$ and $\omega_2$ is a line, called the radical axis of $\omega_1$ and $\omega_2$.

1. If $\omega_1$ and $\omega_2$ intersect at points $A$ and $B$, then their radical axis is $AB$.
2. The radical axis of $\omega_1$ and $\omega_2$ is perpendicular to the line between their centers.

Radical Center:

Fix three circles $\omega_1, \omega_2,$ and $\omega_3$ with centers not all lying on the same line. Then the radical axes of the circles meet at a common point, which is called the radical center of $\omega_1, \omega_2,$ and $\omega_3$.

There are many ways to use power of a point and radical axes! A few of the most important deductions are illustrated below:

In the left figure, $AB, CD, EF$ are radical axes, so they must all meet at a point. In the middle figure, $PC \cdot PD = PA \cdot PB = PE \cdot PF$, and so $CDFE$ is cyclic. In the right figure, $MA^2 = MP \cdot MQ = MB^2$, so $M$ is the midpoint of $AB$.

And now, onto some examples from actual Olympiads!
Example 6. Let $ABC$ be a triangle, and draw isosceles triangles $BCD, CAE, ABF$ externally to $ABC$, with $BC, CA, AB$ as their respective bases. Prove that the lines through $A, B, C$ perpendicular to the lines $EF, FD, DE$, respectively, are concurrent.

Solution: Let $\omega_D$ be the circle with center $D$ and radius $DB = DC$. Define $\omega_E$ and $\omega_F$ analogously. The radical axis of $\omega_D$ and $\omega_E$ passes through their common point ($C$) and is perpendicular to the line joining their centers ($DE$). Thus, the problem is asking us to prove that the radical axes for three circles meet at a point, which we know to be true.

Although it is not related to power of a point, let me also mention that Example 6 is made trivial by the following handy theorem:

Fact 3. Fix points $A, B, C, D, E, F$; let $\ell_1$ be the line through $A$ perpendicular to $EF$, $\ell_2$ be the line through $B$ perpendicular to $FD$, and $\ell_3$ be the line through $C$ perpendicular to $DE$. Then, $\ell_1, \ell_2,$ and $\ell_3$ meet at a point iff $AE^2 - EC^2 + CD^2 - DB^2 + BF^2 - FA^2 = 0$.

Now back to some real-world examples. These both come from the IMO, which means they take a couple different ideas to solve, but power of a point is key in both cases!

Example 7. Let $ABC$ be a triangle with circumcenter $O$. The points $P$ and $Q$ are interior points of the sides $CA$ and $AB$, respectively. Let $K, L$, and $M$ be the midpoints of the segments $BP, CQ$, and $PQ$, respectively, and let $\Gamma$ be the circle passing through $K, L,$ and $M$. Suppose that the line $PQ$ is tangent to the circle $\Gamma$. Prove that $OP = OQ$.

Solution: Since $M$ and $K$ are midpoints of $PQ$ and $BP$, it must be that $MK$ is parallel to and half the length of $QB$. Now,

$$\angle MLK = \angle QMK$$  $(QM$ is tangent to circle $MLK$)

$$= \angle MQA$$  (parallel lines)

$$= \angle AQP.$$  

Similarly, $\angle MKL = \angle APQ$, and so $\triangle MLK$ is similar to $\triangle AQP$. In particular,

$$AQ \cdot MK = AP \cdot ML \implies AQ \cdot (2MK) = AP \cdot (2ML)$$

$$\implies AQ \cdot QB = AP \cdot PC,$$

and hence $P, Q$ have the same power with respect to the circumcircle of $ABC$. But this means that $OP^2 = OQ^2$, and we are done. (Recall the power of $P$ with respect to a circle is $OP^2 - r^2$, where $r$ is the radius of the circle.)

Example 8. Two circles $\Gamma_1$ and $\Gamma_2$ intersect at $M$ and $N$. Let $\ell$ be the common tangent to $\Gamma_1$ and $\Gamma_2$ so that $M$ is closer to $\ell$ than $N$ is. Let $\ell$ touch $\Gamma_1$ at $A$ and $\Gamma_2$ at $B$. Let the line through $M$ parallel to $\ell$ meet the circle $\Gamma_1$ again at $C$ and the circle $\Gamma_2$ again at $D$. Lines $CA$ and $DB$ meet at $E$; lines $AN$ and $CD$ meet at $P$; lines $BN$ and $CD$ meet at $Q$. Show that $EP = EQ$.

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6USAMO 1997, #2.
7IMO 2009, #2.
8IMO 2000, #1.
**Solution:** Extend $NM$ to meet $AB$ at $S$. Then $AS^2 = SM \cdot SN = SB^2$, so $S$ is the midpoint of $AB$. Since $AB$ is parallel to $PQ$, there exists a dilation about $N$ taking $AB$ to $PQ$. This dilation takes $S$ to $M$, and hence $M$ is the midpoint of $PQ$. Also:

$$\angle EAB = \angle ACM \quad \text{(parallel lines)}$$
$$\angle MAB \quad \text{(BA is tangent to circle ACM)}.$$

Similarly, $\angle EBA = \angle MBA$, and hence $\triangle EAB \cong \triangle MAB$ by angle-side-angle. It follows that $EM$ is perpendicular to $AB$, and is therefore perpendicular to $PQ$ as well. Combining this with the fact that $M$ is the midpoint of $PQ$ gives us $\triangle EMP \cong \triangle EMQ$, and the problem is solved. \hfill \square

**Exercises:**

1. Convince yourself that the listed properties for power of a point and radical axis are true. In particular, prove that the radical axes of three circles really must meet at a point.

2. Let $BD$ be the angle bisector of angle $B$ in triangle $ABC$ with $D$ on side $AC$. The circumcircle of triangle $BDC$ meets $AB$ at $E$, while the circumcircle of triangle $ABD$ meets $BC$ at $F$. Prove that $AE = CF$.

3. Draw tangents $OA$ and $OB$ from a point $O$ to a given circle. Through $A$ is drawn a chord $AC$ parallel to $OB$; let $E$ be the second intersection of $OC$ with the circle. Prove that the line $AE$ bisects the segment $OB$.

4. Two equal-radius circles $\omega_1$ and $\omega_2$ are centered at points $O_1$ and $O_2$. A point $X$ is reflected through $O_1$ and $O_2$ to get points $A_1$ and $A_2$. The tangents from $A_1$ to $\omega_1$ touch $\omega_1$ at points $P_1$ and $Q_1$, and the tangents from $A_2$ to $\omega_2$ touch $\omega_2$ at points $P_2$ and $Q_2$. If $P_1Q_1$ and $P_2Q_2$ intersect at $Y$, prove that $Y$ is equidistant from $A_1$ and $A_2$.

5. The altitudes through vertices $A, B, C$ of acute triangle $ABC$ meet the opposite sides at $D, E, F$, respectively. The line through $D$ parallel to $EF$ meets the lines $AC$ and $AB$ at $Q$ and $R$, respectively. The line $EF$ meets $BC$ at $P$. Prove that the circumcircle of triangle $PQR$ passes through the midpoint of $BC$.

6. Let $P$ and $Q$ be points in the plane and let $\omega_1, \omega_2$, and $\omega_3$ be circles passing through both. If $A, B, C, D, E$, and $F$ are points on a line in that order so that $A$ and $D$ lie on $\omega_1$, $B$ and $E$ lie on $\omega_2$, and $C$ and $F$ lie on $\omega_3$, prove that $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.

7. Let $\omega_1$ and $\omega_2$ be concentric circles with $\omega_2$ inside $\omega_1$. Let $ABCD$ be a parallelogram with $B, C, D$ on $\omega_1$ and $A$ on $\omega_2$. If $BA$ intersects $\omega_2$ again at $E$ and $CE$ intersects $\omega_2$ again at $P$, prove that $CD = PD$.

8. Let three circles $\Omega_1, \Omega_2, \Omega_3$, which are non-overlapping and mutually external, be given in the plane. For each point $P$ in the plane, outside the three circles, construct six points $A_1, B_1, A_2, B_2, A_3, B_3$ as follows: For each $i = 1, 2, 3$, $A_i$ and $B_i$ are the two points on the circle $\Gamma_i$ such that the lines $PA_i$ and $PB_i$ are both tangents to $\Gamma_i$. Call the point $P$ *exceptional* if the three lines $A_1B_1, A_2B_2$, and $A_3B_3$ are concurrent. Show that every exceptional point, if it exists, lies on the same circle.
4 Challenge Problems

I will end with a fairly random collection of Olympiad geometry problems for you to practice on, and challenge yourself with. The preceding sections might help you, but they also might not. Good luck – these are hard problems!

1. Two congruent circles $ω_1$ and $ω_2$ intersect at $B$ and $C$. Select a point $A$ on $ω_1$. Let $AB$ and $AC$ intersect $ω_2$ again at $A_1$ and $A_2$. Let $X$ be the midpoint of $BC$. Let $A_1X$ and $A_2X$ intersect $ω_1$ at $P_1$ and $P_2$. Prove that $AP_1 = AP_2$.

2. The circle $k$ is circumscribed about the isosceles triangle $ABC$ (with $AC = BC$) and point $D$ is the midpoint of the arc $AB$. Let $M$ be a point on side $AB$, and let $N$ be the second intersection of the line $DM$ with $k$. Let $P$ and $Q$ be the intersections of the perpendicular bisector of segment $MN$ with $BC$ and $AC$, respectively. Prove that the quadrilateral $CPMQ$ is a parallelogram.

3. A convex quadrilateral $ABCD$ is inscribed in a circle with center $O$. The diagonals $AC$ and $BD$ intersect at $P$. The circumcircles of triangles $ABP$ and $CDP$ intersect again at $Q$. If $O, P,$ and $Q$ are three distinct points, prove that $OQ$ is perpendicular to $PQ$.

4. Let $D$ be a point on side $AC$ of triangle $ABC$. Let $E$ and $F$ be points on the segments $BD$ and $BC$ respectively, such that $∠BAE = ∠CAF$. Let $P$ and $Q$ be points on $BC$ and $BD$ respectively, such that $EP$ and $FQ$ are both parallel to $CD$. Prove that $∠BAP = ∠CAQ$.

5. A circle with center $I$ is inscribed in a quadrilateral $ABCD$ with $∠BAD + ∠ADC > 180°$. A line through $I$ meets $AB$ and $CD$ at points $X$ and $Y$, respectively. Prove that if $IX = IY$, then $AX \cdot DY = BX \cdot CY$.

6. Let $ABC$ be an acute-angled triangle, and let $L$ be the point where the bisector of $∠C$ hits side $AB$. The point $P$ belongs to the segment $CL$ in such a way that $∠APB = 180° - \frac{1}{2} \cdot ∠ACB$. Let $k_1$ and $k_2$ be the circumcircles of $△APC$ and $△BPC$. $BP$ meets $k_1$ again at $Q$, and $AP$ meets $k_2$ again at $R$. The tangent to $k_1$ at $Q$ and the tangent to $k_2$ at $B$ meet at $S$. The tangent to $k_2$ at $R$ and the tangent to $k_1$ at $A$ meet at $T$. Prove that $AS = BT$.

7. Let $ABC$ be a triangle such that $∠A = 90°$ and $∠B < ∠C$. The tangent at $A$ to the circumcircle $ω$ of triangle $ABC$ meets the line $BC$ at $D$. Let $E$ be the reflection of $A$ in the line $BC$, let $X$ be the foot of the perpendicular from $A$ to $BE$, and let $Y$ be the midpoint of the segment $AX$. Let the line $BY$ intersect the circle $ω$ again at $Z$. Prove that the line $BD$ is tangent to the circumcircle of triangle $ADZ$.

8. Two circles $Ω_1$ and $Ω_2$ are contained inside the circle $Ω$, and are tangent to $Ω$ at the distinct points $M$ and $N$, respectively. $Ω_1$ passes through the center of $Ω_2$. The line passing through the two points of intersection of $Ω_1$ and $Ω_2$ meets $Ω$ at $A$ and $B$. The lines $MA$ and $MB$ meet $Ω_1$ at $C$ and $D$, respectively. Prove that $CD$ is tangent to $Ω_2$.

9. The point $M$ is inside the convex quadrilateral $ABCD$, such that $MA = MC$, $∠AMB = ∠MAD + ∠MCD$, and $∠CMD = ∠MCB + ∠MAB$. Prove that $AD \cdot CM = BC \cdot MD$ and $BM \cdot AD = MA \cdot CD$. 

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10. A convex quadrilateral $ABCD$ is given. Prove that there exists a point $P$ inside the quadrilateral such that

$$\angle PAB + \angle PDC = \angle PBC + \angle PDA = \angle PDA + \angle PCB = 90^\circ$$

if and only if the diagonals $AC$ and $BD$ are perpendicular.

11. A non-isosceles triangle $A_1A_2A_3$ has sides $a_1, a_2, a_3$ with side $a_i$ lying opposite vertex $A_i$. Let $M_i$ be the midpoint of side $a_i$, and let $T_i$ be the point where the inscribed circle of triangle $A_1A_2A_3$ touches side $a_i$. Denote by $S_i$ the reflection of the point $T_i$ in the interior angle bisector of angle $A_i$. Prove that the lines $M_1S_1, M_2S_2,$ and $M_3S_3$ are concurrent.

12. Let $AH_1, BH_2, CH_3$ be the altitudes of an acute angled triangle $ABC$. Its incircle touches the sides $BC, CA,$ and $AB$ at $T_1, T_2,$ and $T_3$ respectively. Consider the symmetric images of the lines $H_1H_2, H_2H_3,$ and $H_3H_1$ with respect to the lines $T_1T_2, T_2T_3$ and $T_3T_1$. Prove that these images form a triangle whose vertices lie on the incircle of $ABC$. 
5 Hints

Exercises: Configurations and Directed Angles

2. See Warmup Solutions. Directed angles are probably not a good choice, since equal directed angles do not prove that two regular angles are equal.
3. The other possibility is $\theta = 60^\circ$. Don’t forget to check that your solutions work!
4. Let $Z$ be the intersection of circles $AQR$ and $BRP$. Use directed angles to show it is on circle $CPQ$. Note that $P$ does not necessarily lie on segment $BC$. (Source: Miquel’s Theorem)
5. Use directed angles to calculate $\angle(Q_1Q_2, Q_2Q_3) - \angle(Q_1Q_4, Q_4Q_3)$. (Source: Geometry Unbound)
6. Use sine law. Trig Ceva works, but a cleaner method is to write $PB$ in two ways using $a, h,$ and $\alpha$. If you are not using the condition on $h$, you have not solved the problem. (Source: COMC 1999, B4)
7. Let $T$ denote the intersection of $AM$ and $CD$. Use directed angles and prove $CMDA$ and $KTBA$ are cyclic. Do you see a couple possible configurations? (Source: MOP 1991)
8. You first need to prove the points are cyclic, or at least convex. Otherwise, you will never be able to make your argument precise. Can you prove all the axes of symmetry must meet at a point? (Source: IMO 1998, #1)

Exercises: Power of a Point

1. To prove the radical axis is a line, use coordinates! To prove the radical center exists, let $P$ be the intersection of two radical axes. What is its power with respect to each circle?
2. The angle bisector theorem guarantees $\frac{AD}{BD} = \frac{DG}{BC}$. (Source: Saint Petersburg 1996)
3. Prove that $BO$ is tangent to circle $OEA$. (Source: Geometry Unbound)
4. Let $\omega$ be the circle centered at $X$ with double the radius of $\omega_1$ and $\omega_2$. Then $Y$ is the radical center of $\omega$ and the zero-radius circles centered at $A_1$ and $A_2$.
5. The following are cyclic: $BFEC, DFEM, RBQC$. Show $DQ \cdot DR = PD \cdot MD$. (Source: MOP 1998)
6. Let $R$ be the intersection of $PQ$ and the line. Write down all lengths and use the fact that $R$ has equal power with respect to all three circles. Use directed lengths.
7. $B$ and $C$ have equal power with respect to $\omega_2$. Use this to prove triangles $PCD$ and $BEC$ are similar.
8. Let $Q$ be the point where the three lines meet for some exceptional point, and let $M$ be the midpoint of $PQ$. Prove $M$ is the radical center of $\Gamma_1, \Gamma_2, \Gamma_3$ and that it is constant distance from $P$. (Source: APMO 2009, #3)

Challenge Problems

1. What happens when you reflect about $X$? Let $A'$ be the image of $A$. Then you need to prove $A_1A' = A_2A'$. (Source: Po-Shen Loh)
2. Let $Q'$ be the circumcenter of $\angle MAN$. Prove that $\angle MAQ' = \angle MAQ$, and hence $Q' = Q$. This means $QM = QA = QN$, and the rest should follow. This technique is called working backwards, and it is very useful! (Source: MOP 1998)
3. Prove that $BQOC$ is a cyclic quadrilateral. Directed angles work nicely. (Source: China)
4. Draw triangle $A'EP$ similar to $AQF$, and see what happens. (Source: Crux)
5. By congruent triangles, you should get $\angle AXI$ equals $\angle DYI$ or $\angle CYI$. Only one of these two options is possible. Now chase all angles and look for similar triangles. (Source: Bulgaria 2007, #1)
6. Don’t be scared! There are a lot of points, but the problem is still pure angle-chasing. Let $Z$ be the intersection of $AT$ and $BS$. Then $AZBP$ is a parallelogram, $SBCQ$ is cyclic, and $S,A,C$ are collinear. Now look for some congruent triangles. (Source: Bulgaria 2008, #1)
7. Let $AE$ intersect $BD$ at $P$, and let $AZ$ intersect $BD$ at $M$. Prove that (a) $AZPY$ is cyclic with diameter $AP$, (b) $M$ is the midpoint of $PD$, and then use power of a point. (Source: IMO Short List 1998, G8)
8. Let $E$ be where $AN$ hit $\Omega_2$ again. Prove that $CE$ is tangent to $\Omega_1$ at $C$, then prove $CO_2$ bisects $\angle DCN$. (Source: IMO 1999, #5)
9. Try dissecting $ABCD$ into $ABM, BCM, CDM, DAM$, rescaling some of these triangles, and then putting them together to make a new quadrilateral. Try to make sure the new figure has lengths $CD$ and $\frac{BM}{MA} \cdot AD$ appearing in it somewhere. That way, the complicated product condition simplifies to just showing two lengths are equal. (Source: IMO Short List 1999, G7)
10. A purely synthetic solution to this problem is quite involved, but other approaches work cleanly and fast. Use trig-Ceva to prove that the point exists if the diagonals are perpendicular. (You see how the point has to be constructed, right?) For the other direction, construct a quadrilateral with perpendicular diagonals, and show it is the same as what you started with. (Source: IMO Short List 2008, G6)
11. This was the hardest problem on IMO 1982, but it requires only a straightedge and compass to get the key idea! If you draw the diagram carefully, you should see the sides of $S_1S_2S_3$ are parallel to the sides of $M_1M_2M_3$, which solves the problem. Do you see why? The angle-chasing requires careful book-keeping more than anything else. (Source: IMO 1982, #2)
12. This problem is awfully hard, as befits an IMO #6, but I gave you a big hint already: the triangle formed is precisely $S_1S_2S_3$ from the previous question. To prove this, you need to show that if you take $T_1$, reflect it about the bisector of $\angle A$, then reflect the result about $T_1T_2$, you end up on $H_1H_2$. The composition of these reflections is a rotation centered at the intersection of these two lines. If this intersection point is $X$, prove $\angle BXA = 90^\circ$ and use cyclicness of $BAXH_1$ and $BIXT_1$. (Source: IMO 2000, #6)