1. Let $\omega$ be the circle with centre $O$ and radius $MA$. By lemma 4 it follows that $C$ lies on the polar of $D$ with respect to $\omega$. Therefore $AM^2 = MC \cdot MD$.

2. Let $AD$ intersect the circumcircle $\omega$ of $\triangle ABC$ at $A$ and $F$, and let $E$ be the intersection of $AD$ with $BC$. By lemma 4 $ABFC$ is a harmonic quadrilateral. Then $\frac{BF}{FC} = \frac{AB}{AC}$ and $\frac{\sin(\angle BAF)}{\sin(\angle BAC)} = \frac{\sin(\angle MAC)}{\sin(\angle MAB)}$ where $M$ is the midpoint of $BC$. Hence $\frac{\sin(\angle BAF)}{\sin(\angle BAC)}$ and the result follows.

3. Let $EF$ meet $BC$ at $T$. Let $PQ$ meet $AB$ at $R$. By lemma 2 $P(A, N, R, B)$ is harmonic. Intersecting it with line $EF$, we get $(F, E; Q, T)$ is harmonic. Also $\angle QDT = 90^\circ$ hence by lemma 5 $DQ$ is the angle bisector of $\angle EDF$.

4. Let the diagonals of $ABCD$ intersect at $O$, and let $A'$ be the reflection of $A$ about $M$. Then $O$ is the midpoint of $AC$ hence $BD\parallel A'C$. Then $BD$ and $A'C$ intersect at a point at infinity $S_\infty$. $(D, B; O, S_\infty)$ is harmonic (since $\frac{DO}{OB} = \frac{DS_\infty}{BS_\infty}$), hence the pencil $C(D, B; O, S_\infty)$ is harmonic. The intersection of this pencil with line $AM$ gives four points in harmonic division, hence $(A, A'; K, N)$ is harmonic, hence by problem 1, $MA^2 = MK \times MN$. Since $MP = MA$, it follows that $MP^2 = MK \times MN$. The result follows by Power of a Point.

5. We first complete the diagram. Let $D$ be the point of intersection of $KN$ and $AC$ (wolog $D, A, C$ are collinear in this order). Let $AN$ and $KC$ intersect at $P$, $OP$ intersect $BD$ at $M'$, $BO$ intersect $M'P$ at $H$.

Denote the circumcircle of $AKNC$ by $\omega$. By lemma 6 $DP$ is the polar of $B$ with respect to $\omega$; let $DP$ intersect $\omega$ at $X, Y$. Then $XY$ is the polar of $B$ hence $BX, BY$ are tangent to $\omega$. By $\Omega$ denote the circle with centre $B$ and radius $BX$.

By Brokard’s theorem $\angle OM'D = \angle M'HO = 90^\circ$ so $DM'HO$ is cyclic. By Power of a Point $BD \cdot BM' = BH \cdot BO = BX^2$ (the last equality follows from the fact that $\triangle BXH \sim \triangle BOX$).

Consider the inversion $I$ with respect to $\Omega$. Circles $\Omega$ and $\omega$ are orthogonal, hence $\omega$ is invariant under $I$, so $I(K) = A, I(N) = C, I(A) = K, I(C) = N$. Furthermore $I(M), I(K), I(N)$ are collinear and also $I(M), I(A), I(C)$ are collinear (since $BMKN$ and $BMAC$ are cyclic quadrilaterals). Therefore $I(M)$ is the intersection of $KN$ and $AC$ which is $D$. Hence $BM \cdot BD = BX^2$.

Therefore $BD \cdot BM' = BM \cdot BM$ so $M \equiv M'$ and the result follows.

(There is another solution using spiral similarity. If you have not seen it before, see page 6 of Yufei Zhao’s handout on cyclic quadrilaterals: http://web.mit.edu/yufeiz/www/cyclic_quad.pdf).

6. Let $EF$ intersect $AB$ at $G$. Since $AE, CD, BF$ are concurrent, it follows that $(G, D; A, B)$ is harmonic. Since $FD \perp AM$, it follows that $MA$ is the angle bisector of $\angle GMD$. Then by lemma 2 $\angle AMB = 90^\circ$. Similarly $\angle ANB = 90^\circ$. (This is also Lemma 8 from a list of lemmas by Yufei Zhao: http://web.mit.edu/yufeiz/www/geolemmas.pdf) Therefore $A, N, M, B$ lie on a circle $\omega$ with diameter $AB$. Then $NM = AB \sin(NBM) = AB \sin(90^\circ - \angle IAB - \angle IBA) = AB \sin(\frac{\angle ACB}{2})$.

Let $O$ be the midpoint of $AB$. Then $O$ is the centre of $\omega$ and $\angle NOM = 2 \times \angle NBM = \angle NBM + \angle NAM = \angle ND + \angle MDI$ (since $ANID, IMBD$ are cyclic) $= \angle NDM$ so the cir-
cucircle of \( \triangle NMD \) always passes through \( O \) and the result follows.

7. Let line \( l \) be the common tangent to \( C_1, C_2 \) at \( M \). \( A \) is the pole of line \( BC \) with respect to circle \( C_1 \), and \( A \) lies on line \( MA \), therefore by La Hire’s Theorem, pole of \( MA \) with respect to circle \( C_1 \) lies on line \( BC \). Consider the homothety \( h \) with centre \( M \) which transforms \( C_1 \) into \( C_2 \). Then \( h(B) = E, h(C) = F \) and therefore \( h \) will take line \( BC \) to line \( EF \). Polar relation does not change through homothety (verify this yourself), so \( h \) takes the pole of \( MA \) with respect to \( C_1 \) to the pole of \( MA \) with respect to \( C_2 \); the pole of \( MA \) with respect to \( C_1 \) lies on \( BC \) so the pole \( D' \) of \( MA \) with respect to \( C_2 \) lies on \( EF \). \( D' \) - pole of \( MA \) with respect to \( C_2 \) and so lies on tangents to \( C_2 \) at \( M \) and \( A \). But \( D' \) also lies on \( EF \) and therefore coincides with point \( D \). So \( D \) always lies on tangent to \( C_2 \) at point \( M \) and the result follows.

8. The main idea is that \( AMBN \) is a harmonic quadrilateral. We already have the intersections of \( AC, BD \) and \( AD, BC \). Let us ”complete” the standard picture by drawing \( G \), the intersection of \( BA \) and \( CD \). Let the circumcircle of \( ABM \) intersect \( DC \) again at \( P \). By Power of a Point, \( GD \times GC = GA \times GB = GJ \times GM \), hence \( (G, P; C, D) \) is harmonic by corollary 2. Let \( EF \) intersect \( CD \) at \( P' \), \( AB \) at \( K \) and the circumcircle of \( AMB \) at \( N' \). \( (G, A, K, B) \) is harmonic, the pencil \( E(G, A, K, B) \) is harmonic hence \( (G, D, P', C) \) is harmonic. Therefore \( P \equiv P' \). Then \( P(G, A, K, B) \) is harmonic. Intersecting this pencil with the circumcircle of \( ABM \); we get a harmonic quadrilateral since \( P \) lies on the circumcircle. But the intersections are precisely the points \( M, A, N', B \). Hence \( \frac{MA}{AN'} = \frac{MB}{BN'} \). Therefore \( N' \equiv N \) and the result follows.

9. Consider the homothety \( h \) with centre \( F \) taking \( \omega_2 \) to \( \omega \). (This idea will be explored further during the summer camp lecture in 2 weeks). Then \( h(O_2) = O_1, h(l) \) is a line parallel to \( l \) and tangent to \( \omega \) at \( A' \) where \( A' = h(D) \). Then \( OA' \perp l \) and \( A' \equiv A \). Therefore \( A, D, F \) are collinear. Similarly \( E, D, B \) collinear. We now have our ”standard” cyclic quadrilateral \( AEFB \). Let \( FE \) meet \( AB \) at \( T \) and \( l \) at \( S \). \( D \) lies on the polar of \( T \) (by lemma 6) and since \( l \perp OD \), and \( D \) lies on \( l \), it follows that \( l \) is the polar of \( G \) (by lemma 6). Then \( (T, S; E, F) \) is harmonic by lemma 4, and the pencil \( P(T, S, E, F) \) is harmonic. Intersecting it with line \( O_1O_2 \) we get a harmonic bundle \( (P_\infty, R; O_1, O_2) \) where \( P_\infty \) is a point at infinity and \( R \) is the midpoint of \( O_1O_2 \) hence \( A, O_1, D \) and \( B, O_2, D \) are collinear. The result follows.