Sum of Squares

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The Sum of Squares[1] method (SOS for short) is a powerful technique for solving 3-variable, homogeneous inequalities with equality when $a = b = c$. (It is okay if there are other equality cases as well, but you need to have at least this one). If you have poked around MathLinks, you have probably heard of SOS, but a lot of the descriptions are in Vietnamese, so most people don't know the details. It works as follows:

Sum of Squares:

- Let X be a homogenous expression in terms of a, b, c, and suppose we want to prove $X \geq 0$.
- Write X in the following form:

$$(a-b)^2 \cdot S_c + (b-c)^2 \cdot S_a + (c-a)^2 \cdot S_b.$$

  If X is a rational function and if $X = 0$ when $a = b = c$, this can *always* be done just by following an algorithm.
- Prove $X \geq 0$ from this form, which is often much, much easier than it was originally!

In many ways, you can think of SOS as a cleaner and more powerful version of Muirhead. Like Muirhead, it takes some computation, but the tradeoff is that it solves many problems with zero insight required.

Before going into any more of the details, let's begin with a few basic examples:

Example 1. (AM-GM) *Prove* $a^3 + b^3 + c^3 \geq 3abc$ *for* $a, b, c \geq 0$.

Solution. $a^3 + b^3 + c^3 - 3abc = \frac{a+b+c}{2} \cdot \left((a-b)^2 + (b-c)^2 + (c-a)^2\right) \geq 0.$ □

Example 2. (IMO 1983, #6) *Prove* $a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0$ *for* a, b, c *the sides of a triangle.*

*Based partially on Vietnamese notes from MathLinks.

[1] Note that, despite the name, SOS has very little to do with the useful but ad-hoc technique of writing a positive expression as a positive linear combination of perfect squares. In SOS, the squares are *always* $(a-b)^2, (b-c)^2$, and $(c-a)^2$, and the coefficients need not be positive.

Proof. $a^2b(a-b)+b^2c(b-c)+c^2a(c-a)=\frac{1}{2}\sum_{\text{cyc}}(a-b)^2(a+b-c)(b+c-a)\geq 0.$ □

Example 3. (Schur's inequality) *Prove* $a^3+b^3+c^3+3abc \geq \sum_{\text{cyc}}(a^2b+ab^2)$ *for* $a,b,c\geq 0$.

Solution. $a^3+b^3+c^3+3abc-\sum_{\text{cyc}}(a^2b+ab^2)=\sum_{\text{cyc}}(a-b)^2\cdot\left(\frac{a+b-c}{2}\right)$. Since Schur's inequality is symmetric, we can assume without loss of generality that $a\geq b\geq c$. Then, $a+c-b\geq 0$ and $(c-a)^2=(a-b)^2+(b-c)^2+2(a-b)(b-c)\geq(a-b)^2+(b-c)^2$. Therefore, $\sum_{\text{cyc}}(a-b)^2\cdot\left(\frac{a+b-c}{2}\right)$ is at least

$$\begin{aligned}&(a-b)^2\cdot\left(\frac{a+b-c}{2}+\frac{a+c-b}{2}\right)+(b-c)^2\cdot\left(\frac{b+c-a}{2}+\frac{a+c-b}{2}\right)\\ =\;&(a-b)^2\cdot a+(b-c)^2\cdot c\geq 0.\end{aligned}$$

This completes the proof, but if you do not already know the answer, you should trace back through the argument to see when equality holds (it's not just $a=b=c$). □

AM-GM and the IMO problem become absolutely trivial when written in SOS form. Schur's inequality is more difficult, as befits a subtler inequality, but I will come back a little later to explain where that argument is coming from.

1 Writing inequalities in SOS form

The first step to proving $X\geq 0$ with SOS is to put X in SOS form. In this section, I will describe algorithmically how to do that. As we go, I will illustrate the techniques on the following rather intimidating inequality (USAMO 2003, #5):

$$\sum_{\text{cyc}}\frac{(2a+b+c)^2}{2a^2+(b+c)^2}\leq 8.$$

There will be a certain amount of computation, but significantly less than what you would get from multiplying everything out, and the trade-off is there will be basically no insight required. So without further ado, here is the algorithm:

1. *Group X into one or more terms that are 0 when $a=b=c$.*
   This is usually pretty straightforward and does not require any cleverness. In our example, we have

   $$X=\sum_{\text{cyc}}\left(\frac{8}{3}-\frac{(2a+b+c)^2}{2a^2+(b+c)^2}\right)=\frac{1}{3}\cdot\sum_{\text{cyc}}\frac{4a^2+5b^2+5c^2+10bc-12ab-12ac}{2a^2+(b+c)^2}.$$

   One thing to say is that it is sometimes cleaner to have more than just the three cyclic groups we used here. See Example 4.

2. *Write everything as a multiple of $(a-b),(b-c)$, or $(c-a)$.*
   Let's start with the case of the difference of two monomials, say a^2c-b^2c or a^2-bc. In the first example, the two monomials have equal c exponent, so we can just extract an $a-b$ factor

to get $(a-b)(ac+bc)$. In the second example, we add and subtract an intermediate term and then do the same thing: $a^2 - bc = a^2 - ab + ab - bc = (a-b)a - (c-a)b$.

In general, we write each term as a fraction where the numerator is a polynomial with sum of coefficients equal to 0.[2] We then group the polynomial into differences of monomials and use the above trick. In our example, we have

$$\begin{aligned} X &= \frac{1}{3} \cdot \sum_{\text{cyc}} \frac{2(a^2-ab)+5(b^2-ab)+5(bc-ab)+2(a^2-ac)+5(c^2-ac)+5(bc-ac)}{2a^2+(b+c)^2} \\ &= \frac{1}{3} \cdot \sum_{\text{cyc}} \frac{(a-b)(2a-5b-5c)-(c-a)(2a-5b-5c)}{2a^2+(b+c)^2}. \end{aligned}$$

3. *Group everything by $(a-b), (b-c)$, and $(c-a)$, and ensure the coefficients still vanish when $a=b=c$.*
   The grouping here is pretty straightforward. In our example, we have

$$X = \frac{1}{3} \cdot \sum_{\text{cyc}} (a-b) \cdot \left(\frac{2a-5b-5c}{2a^2+(b+c)^2} - \frac{2b-5a-5c}{2b^2+(c+a)^2} \right).$$

   $\frac{2a-5b-5c}{2a^2+(b+c)^2} - \frac{2b-5a-5c}{2b^2+(c+a)^2}$ vanishes when $a=b=c$, so we're done with this step. In fact, this should happen automatically if you keep everything symmetrical.

   For inequalities that are not fully symmetric though, we might need to do a little more work. For example, suppose we get something like $(a-b)\left(\frac{2a^2-b^2}{a+b}\right)$. The trick here is to add something symmetric to each term: $(a-b)\left(\frac{2a^2-b^2}{a+b} - \frac{a+b+c}{6}\right)$. We have subtracted $((a-b)+(b-c)+(c-a))\left(\frac{a+b+c}{6}\right)$, which does not change the sum, but now each coefficient vanishes when $a=b=c$, as required. (In this case, we could also have subtracted $\frac{a+b}{2}$, which might have been cleaner.)

4. *Write everything as a multiple of $(a-b)^2, (b-c)^2, (c-a)^2, (a-b)(b-c), (b-c)(c-a)$, or $(c-a)(a-b)$.*
   This is exactly the same as Step 2, although it tends to be a little messier because we likely have to clear two (but not three) denominators. In our example, we have

$$\begin{aligned} X &= \frac{1}{3} \cdot \sum_{\text{cyc}} (a-b) \cdot \left(\frac{12a^3-12b^3-9a^2b+9ab^2+9a^2c-9b^2c-3ac^2+3bc^2}{(2a^2+(b+c)^2)(2b^2+(c+a)^2)} \right) \\ &= \sum_{\text{cyc}} (a-b)^2 \cdot \left(\frac{4a^2+4b^2+ab+3ac+3bc-c^2}{(2a^2+(b+c)^2)(2b^2+(c+a)^2)} \right), \end{aligned}$$

   and the inequality has been written in SOS form! In the next section, I will discuss how to complete the proof from this stage.

[2] Even if there are square roots, you can do this with difference of squares: $\sqrt{a} - \sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$. Things might get ugly though!

5. *Replace* $(a-b)(b-c)$ *with* $\frac{1}{2}\left((a-c)^2-(a-b)^2-(b-c)^2\right)$.
   We were somewhat lucky here in that Step 4 immediately put us in SOS form. We could also end up with some $(a-b)(b-c)$ terms. In that case, we just replace $(a-b)(b-c)$ with $\frac{1}{2}\left((a-c)^2-(a-b)^2-(b-c)^2\right)$, and we're done. $(b-c)(c-a)$ and $(c-a)(a-b)$ terms are dealt with similarly.

Using this procedure, you can write just about any X in the form $(a-b)^2\cdot S_c+(b-c)^2\cdot S_a+(c-a)^2\cdot S_b$, and with some practice, you should be able to do it pretty quickly.

Before covering what to do next, let's look at a real problem where the hard part is just getting the inequality into SOS form.

Example 4. (Macedonia Math Olympiad 1999, #5) If a,b,c are positive numbers with $a^2+b^2+c^2=1$, prove that

$$a+b+c+\frac{1}{abc}\geq 4\sqrt{3}.$$

Solution. We first write everything in homogenized form

$$X=a+b+c+\frac{(a^2+b^2+c^2)^2}{abc}-4\sqrt{3(a^2+b^2+c^2)}.$$

This looks pretty terrible, but we can deal with it surprisingly cleanly by splitting it into two parts:

$$\begin{aligned}
& \left(\frac{(a^2+b^2+c^2)^2-3abc(a+b+c)}{abc}\right)+4\left(a+b+c-\sqrt{3(a^2+b^2+c^2)}\right) \\
= & \sum_{\text{cyc}}\left(\frac{a^4+a^2b^2+a^2c^2-3a^2bc}{abc}\right)+4\left(\frac{(a+b+c)^2-3(a^2+b^2+c^2)}{a+b+c+\sqrt{3(a^2+b^2+c^2)}}\right) \\
= & \sum_{\text{cyc}}\left(\frac{(a-b)(a^3+a^2b)-(c-a)(a^3+a^2c)+3(b-c)^2a^2}{2abc}\right)+ \\
& \quad 4\left(\frac{2ab+2bc+2ca-2a^2-2b^2-2c^2}{a+b+c+\sqrt{3(a^2+b^2+c^2)}}\right) \\
= & \sum_{\text{cyc}}(a-b)\cdot\left(\frac{a^3+a^2b-b^3-b^2a+3(a-b)c^2}{2abc}\right)+\sum_{\text{cyc}}\left(\frac{-4(a-b)^2}{a+b+c+\sqrt{3(a^2+b^2+c^2)}}\right) \\
= & \sum_{\text{cyc}}(a-b)^2\cdot\left(\frac{(a+b)^2+3c^2}{2abc}-\frac{4}{a+b+c+\sqrt{3(a^2+b^2+c^2)}}\right) \\
\geq & \sum_{\text{cyc}}(a-b)^2\cdot\left(\frac{(a+b)^2+3c^2}{2abc}-\frac{2}{a+b+c}\right).
\end{aligned}$$

To prove $X\geq 0$, it therefore suffices to show $\frac{(a+b)^2+3c^2}{2abc}\geq\frac{2}{a+b+c}$, but this follows from the fact that $((a+b)^2+3c^2)(a+b+c)\geq(a+b)^2c\geq 4abc$. □

As a fun illustration of just how powerful SOS can be, let's see how it can prove with very little extra effort the following much stronger version of Example 4:

$$17(a+b+c) + \frac{1}{abc} \geq 20\sqrt{3}.$$

By the same argument as before, it suffices to prove $\sum_{\text{cyc}}(a-b)^2 \cdot \left(\frac{(a+b)^2+3c^2}{2abc} - \frac{10}{a+b+c}\right) \geq 0$. However, $a+b+c \geq 3\sqrt[3]{abc}$ and $(a+b)^2+3c^2 \geq 2ab+2ab+3c^2 \geq 3\sqrt[3]{12a^2b^2c^2}$, so that $((a+b)^2+3c^2)(a+b+c) \geq 9\sqrt[3]{12} \cdot abc > 20abc$, and the result follows.

2 Analyzing an inequality in SOS form

With Examples 1, 2, and 4, we got lucky in that S_a, S_b, and S_c were all non-negative. Often, this will not happen. So what do we do in that case? Thankfully, there are some pretty general tools that we can use.[3]

Let a, b, c be real numbers with $a \geq b \geq c$, and suppose one of the following holds:

1. $S_b \geq 0, S_a + S_b \geq 0$, *and* $S_b + S_c \geq 0$,
2. $S_a \geq 0, S_c \geq 0, S_a + 2S_b \geq 0$, *and* $2S_b + S_c \geq 0$,
3. $S_b \geq 0, S_c \geq 0$, *and* $a^2 S_b + b^2 S_a \geq 0$.

Then, $(a-b)^2 \cdot S_c + (b-c)^2 \cdot S_a + (c-a)^2 \cdot S_b \geq 0$.

Proof of Condition 1. $(c-a)^2 = (a-b)^2+(b-c)^2+2(a-b)(b-c) \geq (a-b)^2+(b-c)^2$. Since $S_b \geq 0$, it follows that $(a-b)^2 \cdot S_c + (b-c)^2 \cdot S_a + (c-a)^2 \cdot S_b \geq (a-b)^2(S_c+S_b) + (b-c)^2(S_a+S_b) \geq 0$. □

Proof of Condition 2. $(c-a)^2 = (a-b)^2 + (b-c)^2 + 2(a-b)(b-c) \leq 2(a-b)^2 + 2(b-c)^2$. If $S_b \geq 0$, the claim is trivial. Otherwise, $(a-b)^2 \cdot S_c + (b-c)^2 \cdot S_a + (c-a)^2 \cdot S_b \geq (a-b)^2(S_c + 2S_b) + (b-c)^2(S_a + 2S_b) \geq 0$. □

Proof of Condition 3. Since $a \geq b \geq c$, $\frac{a-c}{b-c} \geq \frac{a}{b}$. As $S_b \geq 0$, we therefore have $(b-c)^2 \cdot S_a + (c-a)^2 \cdot S_b \geq (b-c)^2 \cdot \left(S_a + \frac{a^2}{b^2} \cdot S_b\right) \geq 0$, and the result follows from the fact that $S_c \geq 0$. □

These criteria are not the only ways of proving an inequality once it is in SOS form, but they are easy to use and they come up a lot. If you remember them, you will be able to solve most things (at least symmetric things) that come your way without needing any real insight. For example, if you go back to the proof of Schur's inequality, you will see it is just using Condition 1.

Example 5. (USAMO 2003, #5) Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \leq 8.$$

[3] As SOS is still not very well known, quote this theorem without proof at your own risk on a contest. Fortunately, the proofs for each part are short, so you should be able to reproduce them as needed.

Solution. We put this in SOS form already in the last section, getting $S_c = \frac{4a^2+4b^2+ab+3ac+3bc-c^2}{(2a^2+(b+c)^2)(2b^2+(c+a)^2)}$, and similar values for S_a and S_b. Since everything is symmetric, we can assume $a \geq b \geq c$. Then, clearly $S_b, S_c \geq 0$.

Now, $a^2(2b^2 + a^2 + c^2 + 2ac) \geq b^2(2a^2 + b^2 + c^2 + 2bc)$ because every term on the left-hand side is as big as the corresponding term on the right-hand side, so $a^2 \cdot S_b \geq b^2 \cdot \frac{2a^2+(b+c)^2}{2b^2+(c+a)^2} \cdot S_b$, and

$$\begin{aligned} b^2 S_a + a^2 S_b &\geq b^2 \cdot \left(\frac{4b^2 + 4c^2 + bc + 3ab + 3ac - a^2}{(2b^2 + (c+a)^2)(2c^2 + (a+b)^2)} + \frac{4a^2 + 4c^2 + ac + 3ab + 3bc - b^2}{(2b^2 + (c+a)^2)(2c^2 + (a+b)^2)} \right) \\ &= b^2 \cdot \left(\frac{3a^2 + 3b^2 + 8c^2 + 6ab + 4ac + 4bc}{(2b^2 + (c+a)^2)(2c^2 + (a+b)^2)} \right) \geq 0. \end{aligned}$$

Therefore, the inequality follows from Condition 3. □

Again, you can see that a rather challenging inequality became very weak when placed in basic SOS form. This happens a lot, which is why the SOS method is useful!

3 Cyclic inequalities and extended SOS form

One disadvantage of the basic SOS method is that it does not handle cyclic but asymmetric inequalities very gracefully. In theory, everything is fine, but asymmetric inequalities are messier to put in SOS form, this form is not unique, you need to analyze two separate cases ($a \geq b \geq c$ and $a \leq b \leq c$), and each case can sometimes be quite tricky.

To make things cleaner, it is often useful to work with the following *extended* SOS form:

$$(a-b)^2 \cdot S_c + (b-c)^2 \cdot S_a + (c-a)^2 \cdot S_b + (a-b)(b-c)(c-a) \cdot S.$$

Generally, we write $X = X_0 + X_1$ where X_1 is skew-symmetric (i.e., swapping a and b will negate X_1). We then put X_0 into basic SOS form as discussed above, and we factor $(a-b)(b-c)(c-a)$ from X_1. It is often useful to make X_0 symmetric, but it does not have to be.

Once an inequality is in extended SOS form, we analyze it in much the same way as before. The following criterion is particularly helpful:

Let a, b, c be real numbers and suppose the following holds:

- $S_a \geq 0, S_b \geq 0, S_c \geq 0,$ *and* $27 S_a S_b S_c \geq |S^3(a-b)(b-c)(c-a)|,$

Then, $(a-b)^2 \cdot S_c + (b-c)^2 \cdot S_a + (c-a)^2 \cdot S_b + (a-b)(b-c)(c-a) \cdot S \geq 0.$

Proof. By AM-GM, $(a-b)^2 \cdot S_c + (b-c)^2 \cdot S_a + (c-a)^2 \cdot S_b$ is at least $3\sqrt[3]{(a-b)^2(b-c)^2(c-a)^2 S_a S_b S_c} \geq |S(a-b)(b-c)(c-a)|$. □

There are other things you can do too, but none of them are too fancy. Instead of memorizing more criteria, you should be able to just play around with the inequality directly.

Example 6. (UK TST 2005) *Let x, y, z be positive real numbers satisfying $xyz = 1$. Prove that*

$$\frac{x+3}{(x+1)^2} + \frac{y+3}{(y+1)^2} + \frac{z+3}{(z+1)^2} \geq 3.$$

Solution. Since $xyz = 1$, we can let $x = \frac{a}{b}, y = \frac{b}{c}$, and $z = \frac{c}{a}$. Then, $\frac{x+3}{(x+1)^2} = \frac{\frac{a}{b}+3}{\left(\frac{a}{b}+1\right)^2} = \frac{ab+3b^2}{(a+b)^2}$. Therefore,

$$\begin{aligned} X &= \sum_{\text{cyc}} \left(\frac{ab+3b^2}{(a+b)^2} - 1 \right) \\ &= \frac{1}{2} \cdot \sum_{\text{cyc}} \frac{a^2+b^2-2ab}{(a+b)^2} + \frac{3}{2} \cdot \sum_{\text{cyc}} \frac{b^2-a^2}{(a+b)^2}. \end{aligned}$$

Now,

$$\begin{aligned} \sum_{\text{cyc}} \frac{b^2-a^2}{(a+b)^2} &= \sum_{\text{cyc}} \frac{-(a-b)}{a+b} = \frac{-(a-b)}{a+b} + \frac{(a-b)+(c-a)}{b+c} + \frac{-(c-a)}{c+a} \\ &= (a-b) \cdot \left(\frac{1}{b+c} - \frac{1}{a+b} \right) + (c-a) \cdot \left(\frac{1}{b+c} - \frac{1}{c+a} \right) \\ &= \frac{(a-b)(c-a)}{b+c} \cdot \left(\frac{1}{c+a} - \frac{1}{a+b} \right) = \frac{(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)}. \end{aligned}$$

Therefore, we need to show $\sum_{\text{cyc}} \frac{(a-b)^2}{(a+b)^2} + \frac{3(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)} \geq 0$. However, $S_a, S_b, S_c \geq 0$, and $27 S_a S_b S_c = \frac{27}{(a+b)^2(b+c)^2(c+a)^2} \geq \left| \frac{27(a-b)(b-c)(c-a)}{(a+b)^3(b+c)^3(c+a)^3} \right| = |S^3(a-b)(b-c)(c-a)|$, so this is true. □

This worked out well largely because the asymmetry ended up being only in the numerators. If you have asymmetry in the denominators, you may want to multiply things out first.

4 Parting comments

- Do not fall in love too much with the basic criteria for proving something in SOS form. Other techniques (especially smoothing) can often succeed even when the basic criteria fail!
- Sometimes you will see problems that are just too messy for SOS (e.g. $\sum_{\text{cyc}} \frac{a}{\sqrt{a^2+b+c}} \leq \sqrt{3}$ given $a^2+b^2+c^2 = 3$). If you get stuck on this kind of problem, try using classical techniques to simplify things first (e.g. reduce this example to proving $\sum_{\text{cyc}} \frac{a}{a^2+\frac{(b+c)(a+b+c)}{3}} \leq \frac{3}{a+b+c}$).
- Even Muirhead is not completely supplanted by SOS. For example, $\sum_{\text{cyc}} \frac{1}{a^3+b^3+abc} \leq \frac{1}{abc}$ is much easier to prove with Muirhead than it is with SOS.
- If SOS does not apply directly (e.g. there are 4 variables, things are not homogeneous, etc.), some of the key SOS ideas can still be useful. Separate out squares that vanish in the equality case, factor $(a-b)(b-c)(c-a)$ from skew-symmetric terms, etc.

5 Problems

I recommend using SOS (or at least borrowing the key ideas) to solve these problems. You are welcome to try other techniques as well, although some of these problems could prove quite difficult with classical techniques.

1. *(Full Schur's inequality)* $a^t(a-b)(a-c)+b^t(b-c)(b-a)+c^t(c-a)(c-b) \geq 0$ for $a,b,c,t \geq 0$.
2. *(Nesbitt's inequality)* $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}$ for $a,b,c > 0$.
3. *(Somewhere near Russia, 2008)* $\frac{3a-1}{1-a^2}+\frac{3b-1}{1-b^2}+\frac{3c-1}{1-c^2} \geq 0$ for $a,b,c > 0$ satisfying $a+b+c=1$.
4. *(CMO 2008, #3)* $\frac{a-bc}{a+bc}+\frac{b-ca}{b+ca}+\frac{c-ab}{c+ab} \leq \frac{3}{2}$ for $a,b,c > 0$ satisfying $a+b+c=1$.
5. *(IMO 2000, #2)* $\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \leq 1$ for $a,b,c > 0$ satisfying $abc=1$.
6. *(Iran 1996)* $\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2} \geq \frac{9}{4(ab+bc+ca)}$ for $a,b,c > 0$.
7. *(Japan MO 1997, #2)* $\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(c+a-b)^2}{(c+a)^2+b^2}+\frac{(a+b-c)^2}{(a+b)^2+c^2} \geq \frac{3}{5}$ for $a,b,c > 0$.
8. *(Japan MO 2004, #4)* $\frac{1+a}{1-a}+\frac{1+b}{1-b}+\frac{1+c}{1-c} \leq 2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)$ for $a,b,c > 0$ satisfying $a+b+c=1$.
9. *(Vietnam TST 2006, #4)* $(a+b+c)\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 6\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$ for a,b,c the sides of a triangle.
10. $\frac{4a}{a+b}+\frac{4b}{b+c}+\frac{4c}{c+a}+\frac{ab^2+bc^2+ca^2+abc}{a^2b+b^2c+c^2a+abc} \geq 7$ for $a,b,c > 0$.
11. $\frac{a^4}{a^3+b^3}+\frac{b^4}{b^3+c^3}+\frac{c^4}{a^3+b^3} \geq \frac{a+b+c}{2}$ for $a,b,c > 0$.
12. *(Balkan MO 2005, #3)* $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \geq a+b+c+\frac{4(a-b)^2}{a+b+c}$ for $a,b,c > 0$.
13. $\frac{2a}{b^3+c^3}+\frac{2b}{c^3+a^3}+\frac{2c}{a^3+b^3} \leq \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$ for a,b,c the sides of an acute triangle.
14. *(IMO Short list 2006, A5)* $\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}}+\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} \geq 0$ for a,b,c the sides of a triangle.
15. *(IMO 2006, #3)* Determine the least real number M such that $|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)| \leq M(a^2+b^2+c^2)^2$ for all a,b,c.
16. *(IMO Short list 2008, A7)* $\frac{(a-b)(a-c)}{a+b+c}+\frac{(b-c)(b-d)}{b+c+d}+\frac{(c-d)(c-a)}{c+d+a}+\frac{(d-a)(d-b)}{d+a+b} \geq 0$ for $a,b,c,d > 0$.
17. $ab^2+bc^2+ca^2 \leq 2+abc$ for $a,b,c \geq 0$ satisfying $a^2+b^2+c^2=3$.

6 Problems in SOS form

To help you check your work, I have written each of the problems in SOS form here.

1. $\frac{1}{2}\sum_{\text{cyc}}(a-b)^2 \cdot (a^t + b^t - c^t) \geq 0$.
2. $\frac{1}{2}\sum_{\text{cyc}}(a-b)^2 \cdot \left(\frac{1}{(a+c)(b+c)}\right) \geq 0$.
3. $\sum_{\text{cyc}}(a-b)^2 \cdot \frac{(a-b)^2(a+b)}{(1-a^2)(1-b^2)} \geq 0$.
4. $\frac{1}{2}\sum_{\text{cyc}}(a-b)^2 \cdot \frac{c}{(a+b)(b+c)(c+a)} \geq 0$. This one is also clean with Muirhead.
5. Just rewrite it as Schur's inequality.
6. $\frac{1}{4(ab+bc+ca)} \cdot \sum_{\text{cyc}}(a-b)^2 \cdot \left(\frac{3ab+bc+ca-c^2}{(a+c)^2(b+c)^2}\right) \geq 0$.
7. $\frac{2}{5}\sum_{\text{cyc}}(a-b)^2 \cdot \left(\frac{3a^2+3b^2-c^2+2ac+2bc}{((b+c)^2+a^2)((c+a)^2+b^2)}\right) \geq 0$.
8. $\sum_{\text{cyc}}(a-b)^2 \cdot \left(\frac{1}{ab} - \frac{1}{(a+c)(b+c)}\right) - \frac{(a-b)(b-c)(c-a)}{abc} \geq 0$. The criterion I gave for extended SOS doesn't work, but there is still an easy proof from here.
9. $\sum_{\text{cyc}}(a-b)^2 \cdot \left(\frac{1}{ab} - \frac{3}{(a+c)(b+c)}\right) \geq 0$.
10. $\frac{(a-b)^2(b-c)^2(c-a)^2}{(a+b)(b+c)(c+a)(a^2b+b^2c+c^2a+abc)} \geq 0$.
11. $\frac{1}{4}\sum_{\text{cyc}}(a-b)^2 \cdot \left(\frac{a^2+b^2+ab}{a^3+b^3}\right) - (a-b)(b-c)(c-a) \cdot \left(\frac{a^2b^2+b^2c^2+c^2a^2}{2(a^2+b^2-ab)(b^2+c^2-bc)(c^2+a^2-ca)}\right) \geq 0$.
12. $\sum_{\text{cyc}}(a-b)^2 \cdot \frac{1}{b} \geq (a-b)^2 \cdot \frac{4}{a+b+c}$.
13. $\frac{1}{a^2b^2c^2(a^3+b^3)(b^3+c^3)(c^3+a^3)} \cdot \sum_{\text{cyc}}(a^6-b^6)(a-b)(c^5a+c^5b-a^2b^2c^2) \geq 0$.
14. $\sum_{\text{cyc}} \frac{(\sqrt{c}-\sqrt{a})(\sqrt{c}-\sqrt{b})}{(\sqrt{a}+\sqrt{b}-\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c}+\sqrt{a+b-c})} \geq 0$. It's convenient to stop simplifying here.
15. This is not a real SOS problem since $a=b=c$ is not an equality case, but the key first step is the hopefully now familiar idea of writing the left-hand side as $|(a-b)(b-c)(c-a)(a+b+c)|$.
16. This is not standard SOS since there are 4 variables, but the idea is the same. $(a-c)^2 \cdot \frac{S+a+c}{(S-b)(S-d)} + (b-d)^2 \cdot \frac{S+b+d}{(S-a)(S-c)} \geq 3(a-c)(b-d) \cdot \frac{(a+c)(S-a)(S-c)-(b+d)(S-b)(S-d)}{(S-a)(S-b)(S-c)(S-d)}$.
17. After normalizing: $\sum_{\text{cyc}}(a^2-b^2)^2 \cdot (4a^2+4b^2+c^2) \geq 27(a^2-b^2)(b^2-c^2)(c^2-a^2) - 108abc(a-b)(b-c)(c-a)$. Beware: this is very tight!